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Linear logic, coherence and dinaturality

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Abstract

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A general coherence theorem for monoidal closed structures is obtained by modifying the logical approach to coherence questions, due to Lambek [1969, 1990] by making use of linear logic. Lincar logic, introduced by Girard, has many advantages which are of use in studying coherence. Most notably, its resource-sensitive nature makes it ideal for studying monoidal closed structures. The logical approach is also modified by using natural deduction rather than sequent calculus. The natural deduction system in question is proof nets, also introduced by Girard. Proof nets have several important properties which are exploited to prove the coherence theorem. In particular, the cut elimination procedure is confluent and strongly normalizing.

The approach to coherence is to define a general structure, the *autonomous deductive system*, for defining many theories of monoidal closed categories. An autonomous deductive system is a deductive system with several added features, which are suggested by the properties of proof nets. It is then possible to give a straightforward criterion for whether a given theory of monoidal closed categories, specified by an autonomous deductive system, is coherent.

Finally, a relationship is established between coherence and the composition problem for dinatural transformations. Thus, the dinatural approach to modelling polymorphic types, due to Bainbridge et al. [1990], can be extended to linear polymorphism.

0. Introduction

While the subject of coherence arose originally in topology, it has been most heavily influenced by logical principles. This approach began with the fundamental work of Lambek in [36]. Lambek's idea was to view the various theories of monoidal closed categories as deductive systems. Thus, the operations specified by the theory can be viewed as inference rules. Given this viewpoint, logical methods, most importantly cut

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elimination, can be put to use. Following Lambek's initial work, Kelly and Mac Lane [34] obtained a partial coherence theorem for symmetric monoidal closed (hereafter called autonomous) categories. Subsequently, a great deal has been accomplished via this approach; see [30, 34, 41].

The essential point of a coherence theorem is to show that a certain class of diagrams, arising in the theory of closed categories in question, commutes. Inductive approaches to the question tend to fail because any given edge of the diagram may itself be a composite of indeterminately many more basic morphisms. But the cut elimination theorem in this setting says that every such morphism is equal to a morphism not built up from more basic components. After application of this result, an inductive approach can be put to use.

It is important to note that one of the consequences of Lambek's insight is a strengthening of the notion of cut elimination. Previous cut elimination theorems said that, for every deduction derived with cut, there is a corresponding cut-free deduction. Lambek's version says that, under the equivalence relation generated by the equations of the theory, every deduction is equal to a cut-free deduction.

In this paper, expanding on the work begun in [4], we study coherence theory from the viewpoint of linear logic, defined by Girard in [18]. Linear logic can be seen as a refinement of classical and intuitionistic logic in the sense that both can be embedded into linear logic. One of the advantages of linear logic is that it provides greater control over the structural rules, contraction and weakening. This is crucial for the study of coherence.

Most previous uses of the Lambek approach have focussed on sequent calculus. In this paper, we prefer to use an approach based on natural deduction. Indeed, the notions of natural deduction and coherence share many of the same properties. Girard, in [26], analyzes natural deduction and describes it as a more intrinsic system than sequent calculus. Deductions in sequent calculus, which differ only in irrelevant choices of the order in which the rules are applied, are equated in natural deduction. Analogously, the equations which comprise the theory of autonomous categories can be viewed as equating deductions with redundancies to reductions without these redundancies. In both cases, the notion of an equivalence relation on proofs is crucial. Ultimately, it is hoped that the study of coherence will provide an abstract notion of natural deduction for more general logical systems.

Proof nets, a natural deduction system for linear logic, have many important features which make them especially relevant to the study of coherence. Most notably, the cut elimination process is confluent and strongly normalizing. A consequence of this is that cut-free proof nets provide unique normal forms. The analogy is to simply typed lambda calculus and beta reduction. The uniqueness of normal form is of great use in obtaining coherence theorems.

Proof nets are built inductively from four types of links, one of which is the axiom link. In this paper, we will show that they correspond precisely to a notion in the original Kelly–Mac Lane paper, the Kelly–Mac Lane graph. Both are pairings of the variables in the sequent, and both satisfy the same variance condition. A consequence of this correspondence is that certain notions studied in the Kelly–Mac Lane paper can be shown to be equivalent to properties of proof nets. Most importantly, the notion of compatibility is equivalent to the acyclicity condition of proof nets.

Working with proof nets makes it possible to obtain a general, or modular, coherence theorem. By this is meant that we have obtained a theorem which applies to several, in fact, infinitely many, monoidal closed theories. One begins with a base system, multiplicative linear logic, and then is allowed to add on certain extra structure. This system is the *autonomous deductive system* (ADS). To check that the resulting theory satisfies the various coherence properties, one need only check that the properties hold on the additional structure. This is a consequence of the modularity of proof nets themselves, especially the version introduced by Danos and Regnier in [12]. One may add to their proof nets nonlogical axioms (see [18]), additional inference rules (see [15]), and quantifiers (see [21, 24]).

Coherence issues have been of great interest in computer science recently following the work of [8, 10, 43]. In certain type systems, most notably those containing a notion of inheritance, a semantics is defined which assigns to certain judgements, typically the inheritance judgements, a canonical morphism. In the case of inheritance, a judgement of the inheritance relation

$$e\colon s \leqslant t$$

determines semantically a morphism,

$$|e|\colon |s| \to |t|,$$

which in the simpler cases can be thought of as type inclusion. But the semantic definition is inductive, induction being on the length of a specific derivation of this typing judgement. So, it needs to be shown that the semantics does not depend on the specific typing judgement chosen. As before, this necessitates defining an equivalence of deductions. Different derivations of the same judgement must yield equal morphisms in the category. This avoids what Reynolds refers to as "semantic ambiguity".

The analogy to "classical" categorical coherence suggests an interesting interpretation of the Kelly and Mac Lane result. The term assigned to the inheritance judgement behaves in much the same way as the Kelly–Mac Lane graph. That is, it classifies which diagrams are to commute. This suggests that Kelly–Mac Lane graphs can be viewed as a nonstandard form of syntax. Given the connection to axiom links, this is not surprising. Traditional syntax, i.e. lambda terms for cartesian-closed categories, are generally obtained using natural deduction, under the Curry–Howard isomorphism, see [38]. Since Kelly–Mac Lane graphs are also axiom links, the graph encodes the deduction in much the same way as a lambda term encodes a deduction. This follows from the fact that the cut-free proof net of a sequent is uniquely determined by the sequent and its axiom links. This approach may be especially useful given that Kelly–Mac Lane graphs seem to have many properties desirable for the study of parallelism. This is suggested by Lafont's interaction nets; see [35].

Finally, a connection between coherence and dinaturality is established. Dinatural transformations were originally defined in [13]. They are the appropriate notion of transformation for multivariant functors. Following the work of [1], they can be used in the study of polymorphic type theory. Since constant types are modelled by objects of a category, it seems natural to model variable types as functors, and judgements as natural transformations. However, this is problematic because it is possible for the same type variable to appear both covariantly and contravariantly in a type expression.

The solution proposed in [1] is to replace functors with a special class of multivariant functors. Then typing judgements become dinatural. However, for this approach to work, it must be the case that these dinaturals compose, a property not always true of dinatural transformations. Using the previous work on coherence, we are able to show that, in certain cases, the dinatural transformations corresponding to typing judgements do compose. In fact, this property is shown to be equivalent to coherence, that is, that an ADS has a compositional dinatural calculus if and only if it is coherent.

In this way, we can develop a notion of linear polymorphism, as well as giving a large class of models for it.

1. Kelly–Mac Lane coherence

Kelly and Mac Lane [34] studied the coherence question for autonomous (symmetric monoidal closed) categories. For a complete definition, including all the equations that an autonomous category must satisfy, see [34]. They introduce several new concepts into the study of coherence. One of the problems to be solved for a coherence theorem is to specify the diagrams which do commute in the theory. It is only in the simplest monoidal theories that all diagrams will commute. As a partial solution to this question, they introduce the notion of graph. Every morphism specified by the theory is assigned a unique Kelly–Mac Lane graph. Kelly–Mac Lane graphs can be thought of as classifying morphisms in the sense that a pair of morphisms will be equal only if they have the same graph. Unfortunately, there are some cases in which graphs are insufficient to classify morphisms, as will be seen below.

Kelly and Mac Lane begin by defining the *allowable morphisms*. These are the morphisms specified by the theory. In our terminology, they are the deductions. So, the presentation of a morphism will be a deduction in the appropriate deductive system.

Kelly–Mac Lane graphs are then defined by the following procedure. First, define *shapes* to be the formal objects of the theory, built up from variables, I, $\otimes$, and $\multimap$. The allowable morphisms are then seen as morphisms between shapes.

Assign to each shape a variable set, as follows. (A variable set is a sequence of $+$'s and $-$'s.) The variable set is denoted as $v(\boldsymbol{T})$.

$$v(\boldsymbol{I})=\emptyset,$$

$$v(\boldsymbol{X})=\{+\},$$

$$v(\boldsymbol{T}\otimes\boldsymbol{S})=v(\boldsymbol{T})\coprod v(\boldsymbol{S}),$$

$$v(\boldsymbol{T}\multimap\boldsymbol{S})=v(\boldsymbol{T})^{\mathrm{op}}\coprod v(\boldsymbol{S}).$$

In the above, $v(\boldsymbol{T})^{\mathrm{op}}$ is $v(\boldsymbol{T})$ with $+$'s changed to $-$'s and vice versa. $\coprod$ stands for concatenation of lists. A graph is then defined to be a fixed-point free pairing of the variables in $\boldsymbol{T}$ and $\boldsymbol{S}$, such that paired elements have opposite variance in $v(\boldsymbol{T})^{\mathrm{op}}\coprod v(\boldsymbol{S})$.

Example. Two examples are shown in Fig. 1.

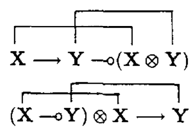

Fig. 1.

It can then be shown that every allowable morphism has a graph.

One of the key notions of this paper is that of compatibility of graphs. We will later show that this concept of [34] is equivalent to the acyclicity condition for proof nets.

Two graphs $f\colon \boldsymbol{T}\to\boldsymbol{S}$ and $g\colon\boldsymbol{S}\to\boldsymbol{R}$ are *compatible* if there does not exist a sequence $X_1, X_2, \ldots, X_{2r}$ of variables in $\boldsymbol{S}$ such that X_{2i-1} and X_{2i} are paired under f and X_{2i}, X_{2i+1} are paired by g, and X_{2r}, X_1 are also paired by g. The following are examples of incompatible graphs. Imagine that the domain and codomain are written vertically, and that the connectives are suppressed, as this makes the loops easier to see.

Example. The first example is obtained by considering the evident graphs (see Fig. 2)

$$f\colon \boldsymbol{Y}\to\boldsymbol{Y}\otimes(\boldsymbol{X}\multimap\boldsymbol{X}) \quad\text{and}\quad g\colon \mathbf{Y}\otimes(\mathbf{X}\multimap\mathbf{X})\to\mathbf{Y}.$$

For the second example, suppose we have the two graphs shown in Fig. 3a. Composing them yields the incompatibility shown in Fig. 3b.

Graphs can be composed to form a category, as follows. Graph composition amounts to joining together the line segments comprising the graph, in the obvious way.

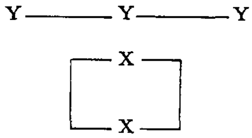

Fig. 2.

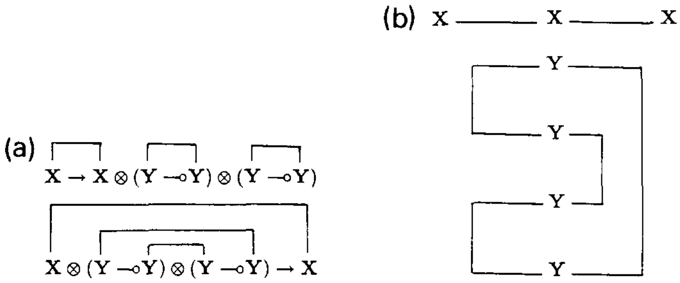

Fig. 3.

With this in mind, the definition of compatibility says that no loops are formed when you compose the two graphs. Incompatible graphs can still be composed by ignoring any loops which do arise. So, for example, the composition of the two morphisms in our second example of incompatibility is just

$$X \text{———} X.$$

In this way, we get a category, denoted as $\boldsymbol{G}$.

The objects of $\boldsymbol{G}$ will be signed sets. The morphisms will be Kelly–Mac Lane graphs, with composition as described above.

The two main theorems of the Kelly–Mac Lane paper are as follows.

Theorem 1.1. *If $f: \boldsymbol{T} \to \boldsymbol{S}$ and $g: \boldsymbol{S} \to \boldsymbol{R}$ are allowable, then they are compatible.*

Theorem 1.2. *If $f, g: \boldsymbol{T} \to \boldsymbol{S}$ are allowable and have the same graph, and the shapes $\boldsymbol{T}, \boldsymbol{S}$ do not contain subshapes of the form $\boldsymbol{R} \multimap \boldsymbol{I}$, then $f = g$.*

Note that Kelly–Mac Lane graphs can also be defined for the theory of $*$-autonomous categories. The only modification necessary is that one defines

$$v(\boldsymbol{T}^{\perp}) = (v(\boldsymbol{T}))^{\mathrm{op}}.$$

From the viewpoint of linear logic, shapes are propositions in the intuitionistic fragment of multiplicative linear logic, (mLL). Allowable morphisms are valid deductions; coherence conditions form an equivalence relation on deductions of the same graph.

Most importantly, the linkings of Kelly–Mac Lane graphs correspond to the axiom links of a proof structure. Incompatibility of graphs corresponds to a failure of the long-trip condition of Girard's proof nets. In terms of Danos–Regnier proof nets, this is the acyclicity condition.

We will define this correspondence precisely later in the paper.

It is important to note that the Kelly–Mac Lane result is only partial in the sense that it fails in the case when there are subshapes of the form $\boldsymbol{R} \multimap \boldsymbol{I}$. A complete solution to coherence for autonomous categories has recently been obtained by Jay using his "languages for monoidal categories" see [29, 30].

It seems that the failure for graphs to classify deductions in this case is closely related to the problems with extending proof nets to handle the theory of multiplicative linear logic enriched with units. Since our approach to coherence uses proof nets, we will only discuss theories without units. So, our theory of autonomous categories will be the usual theory without units, and, analogously, for the other theories we discuss.

This has the obvious disadvantage that the theorem we get for any specific theory will not be the best possible. However, we can obtain a general theory which applies to infinitely many cases at once.

2. Danos–Regnier proof nets

Girard, in [18], introduced proof nets. They are an appropriate version of natural deduction for mLL. They have several important properties which we list below. For a further discussion, see [26].

An alternate version was subsequently introduced by Danos and Regnier [12]. While their system is equivalent to the original proof nets, it has several additional features which make it more useful for this paper. One of these features is their modularity. To the base system can be added additional features such as inference rules. In particular, the system can be extended to handle the theory mLL+(MIX). This was first noted by Fleury and Rétoré [15]. This will be the theory we will use the most.

The MIX rule will be written here as

$$\frac{\Gamma \vdash \qquad \vdash \Delta}{\Gamma \vdash \Delta}.$$

In the presence of negation, this version is equivalent to the original version presented in [15]. We write it in this way because it is easier to model this version in a category. This will be evident when we define models of ADS.

The construction proceeds in the following way. A *proof structure* is built inductively from the four types of links shown in Fig. 4.

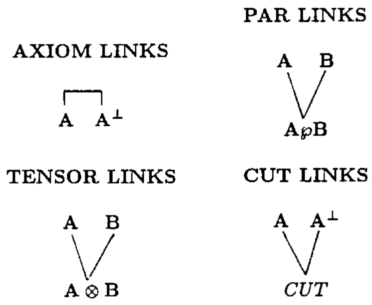

Fig. 4.

So, a proof structure is a graph (not as in the Kelly–Mac Lane sense, but the usual sense). The correctness criteria for mLL and mLL + (MIX) are given by associating with a proof structure a family of subgraphs. This family is obtained by removing one of the two edges from each par link. Thus, there are 2^n subgraphs, where n is the number of par links. The criterion for mLL is then that each subgraph must be acyclic and connected. For mLL + (MIX), the criterion is that each subgraph must be acyclic.

The fact that proof nets provide a correct natural deduction system for mLL or for mLL + (MIX) is indicated by the following two theorems. They were proved in the mLL case by Girard in [18]. They can be extended to the larger theory mLL + (MIX); see [15].

Theorem 2.1 (Girard [18]). *There is a canonical translation procedure which takes sequent calculus deductions in mLL to proof structures. If the deduction is correct, then that deduction is taken to a proof net.*

Theorem 2.2 (Girard [18]). *Given a proof net, there is a sequent calculus proof mapped to it under the translation procedure.*

Proof structures which are valid in mLL + (MIX) will be called generalized proof nets. The mix rule acts on proof structures by disjoint union. As pointed out in [15], this can be thought of as the slogan "lack of communication is an instance of communication".

Note that the one portion of the multiplicative fragment which is not incorporated in proof nets is the units. As noted above, this seems related to the problems with extending Kelly–Mac Lane style coherence to the full theory of autonomous categories with units.

3. Cut elimination

As previously remarked, the cut elimination procedure is especially important for proving coherence theorems. For proof nets, it is accomplished by the following procedure. The advantage of this procedure is that it is local in nature, so that each cut can be eliminated independently, as shown in Fig. 5.

The following theorem is due to Girard.

Theorem 3.1. *The theory mLL satisfies cut elimination. The cut elimination process for proof nets is confluent and strongly normalizing. Given a correct sequent in mLL, its cut-free proof net is uniquely determined.*

For a proof of these results, see [18]. All of the above results are easily extended to the theory mLL+(MIX).

The reason that the mix rule is so important to this work is that the correctness criterion for it corresponds exactly to the notion of compatibility of graphs. We will give a construction which transforms a cycle in a proof structure into two incompatible graphs, and conversely. The theory mLL has the additional criterion of connectedness which is not relevant for our purposes. So, if we wish to characterize exactly those theories of closed categories for which compatibility holds, we must work in the slightly larger system mLL+(MIX).

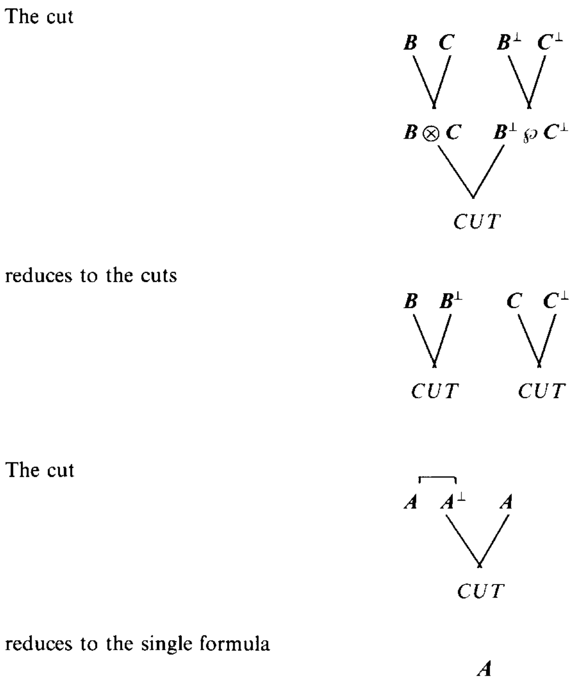

Fig. 5.

4. Autonomous deductive systems

We now introduce the main definition of the paper.

Autonomous deductive systems (ADS) are a general framework for defining the theories of various closed categories. They are an extension of Lambek's notion of deductive system (see [36, 37]) to linear logic sequents. As in any deductive system, there are deductive rules and equality rules. We impose additional restrictions on these rules which make the system behave more like linear logic. There are two types of ADS, the intuitionistic and the classical. Their syntax is defined as follows:

Intuitionistic case

Connectives are: $\otimes$, $\multimap$

Classical case

Connectives are: $\otimes$, $⅋$

In the classical case, the operator $(-)^{\perp}$ is built into the syntax using Girard's notion of De Morgan's laws; see [18]. So, then variables are of the form A or $A^{\perp}$. Negation is then extended to nonatomic formulas using the De Morgan rules. However, in this paper, we may write expressions such as $(A^{\perp} \otimes B^{\perp})^{\perp}$ with the obvious meaning. (In previous papers studying closed categories and coherence issues, the primitive connectives were always $\otimes$, $(-)^{\perp}$.)

Propositions are built up from variables and the connectives. Sequents are of the form

intuitionistic case: $\Gamma \vdash A$

classical case: $\Gamma \vdash \Delta$

Γ, Δ are finite lists. A is a single proposition.

It is further stipulated that to each sequent is associated a Kelly–Mac Lane graph. Each rule of inference must come equipped with a rule for assigning a graph to the conclusion, given the graphs of the premises. The inference rules are simply those of the appropriate fragment of linear logic. These rules may be found in the classical case [18] and in the intuitionistic case [25]. It is easy to see how they operate on graphs. For example, the cut rule acts on graphs by graph composition, which is described previously. To model the right tensor rule, take the disjoint union of the graphs of the premises, as shown in Fig. 6.

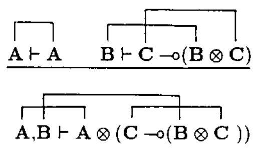

Fig. 6.

The Kelly–Mac Lane graph associated with the sequent should be thought of as representing the deduction of that sequent. In the two theories mLL and mLL + (MIX), once a sequent and its axiom links are specified, then the cut-free proof net is uniquely determined. The cut-free net is essentially the subformula tree of each formula in the sequent, together with the axiom links.

In general, a net will consist of four layers, the topmost being the axiom links. Beneath this is the list of atoms appearing in the deduction. In the third layer, these atoms are combined using the connective links, tensor and par. Finally, at the bottom appear the **CUT** links. The following example of a net illustrates this.

Example. See Fig. 7.

If the net is cut-free, only the first three layers will appear. In this case the Kelly–Mac Lane graph of the sequent will be precisely the axiom links. If there are cuts, then the cut elimination process induces a permutation on the atoms which determines the axiom links of the resulting cut-free net. This process is described in detail in [20].

So, in particular, note that graph composition can be described on the nets themselves.

5. Restriction on additional axioms

As this is a general system, you are allowed to add in additional, nonlogical axioms, subject to the following restriction.

Any added nonlogical axiom must come equipped with a graph, and any axiom which has the same variable appearing more than twice must be part of an axiom scheme in which each variable appears exactly twice. For example, if one wishes to have an added axiom of the form shown in Fig. 8(a), then it must appear as a substitution instance of the more general axiom given in Fig. 8(b).

So, the axiom must be added in the latter form. In Lambek's terminology (see [36]) the axiom must be added in its greatest possible generality. Then the less general form

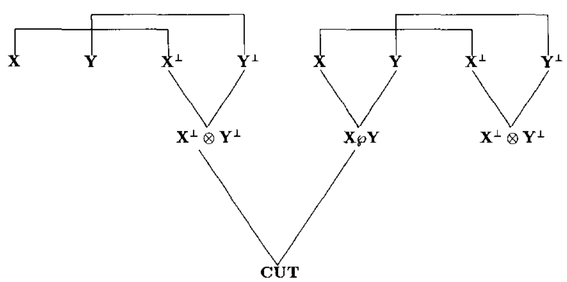

Fig. 7.

(a) $A \otimes (A^{\perp} \otimes A) \longrightarrow A$

(b) $B \otimes (A^{\perp} \otimes A) \longrightarrow B$

Fig. 8.

can be derived. So, each added axiom is really an axiom scheme, and one is allowed to substitute into the axiom scheme as above.

This restriction is intended to match the intuition that if two variables are not linked by the axiom's graph, then it should be possible to substitute different terms for them. Perhaps, it is best to think of the links themselves as the real variable, and the letters A, etc., as placeholders.

This also furthers the intuition that the Kelly–Mac Lane graph provides an abstract notion of syntax.

Before introducing equality rules, we will need the notion of *characteristic* of a deduction.

6. The characteristic of a deduction

Definition 6.1. Given a deduction $\mathscr{D}$, we define its characteristic, a nonnegative integer, $\chi(\mathscr{D})$, by induction on the complexity of $\mathscr{D}$.

If $\mathscr{D}$ consists of an axiom, either logical or nonlogical, then $\chi(\mathscr{D})=0$.

If $\mathscr{D}$ is obtained by the application of the TENSOR or MIX rules, from deductions $\mathscr{D}_1$ and $\mathscr{D}_2$, then $\chi(\mathscr{D})=\chi(\mathscr{D}_1)+\chi(\mathscr{D}_2)$.

If $\mathscr{D}$ is obtained by the application of a PAR rule to $\mathscr{D}_1$, then $\chi(\mathscr{D})=\chi(\mathscr{D}_1)$.

If $\mathscr{D}$ is obtained by the application of a cut rule from deductions $\mathscr{D}_1$ and $\mathscr{D}_2$, then $\chi(\mathscr{D})$ is calculated as follows. Remember that there is an inductive procedure for determining the Kelly–Mac Lane graph of a deduction and the procedure at a cut rule is graph composition. So, $\mathscr{D}_1$ and $\mathscr{D}_2$ each have Kelly–Mac Lane graphs and the graph for $\mathscr{D}$ is obtained by composing them. Let n be the number of instances of incompatibility obtained by this composition, i.e. the number of loops formed. Then

$$\chi(\mathscr{D})=\chi(\mathscr{D}_1)+\chi(\mathscr{D}_2)+n.$$

What the characteristic does is to count the number of loops formed by graph composition during the course of a deduction. Note, in particular, that a cut-free deduction always has characteristic 0. It will be a consequence of Theorem 9.3 that any correct deduction in mLL or mLL+(MIX) also has characteristic 0.

Just as graph composition can be described on the nets themselves, the characteristic of a deduction can also be defined on the proof structure assigned to that deduction. It can be shown that the characteristic when defined on nets is equal to the characteristic of the corresponding sequentialization, regardless of the particular sequentialization chosen.

7. Equality rules

We wish to allow that certain deductions can be equated, but with some restrictions. Two deductions are equatable if they are deductions of the same sequent and satisfy the following two conditions. First, the two deductions should assign the same graph to the sequent, and, second, they must have equal characteristic. So, in particular, a deduction without an instance of incompatibility cannot be equated with a deduction with an instance of incompatibility. As an example of when two deductions cannot be equated, consider the ADS with the following two additional axioms (each with the obvious graph):

$$A \vdash A \otimes B \otimes B^{\perp},$$

$$A \otimes B \otimes B^{\perp} \vdash A.$$

In this ADS, there is a deduction of the following form:

$$\frac{A \vdash A \otimes B \otimes B^{\perp} \qquad A \otimes B \otimes B^{\perp} \vdash A}{A \vdash A}.$$

While this deduction has the same Kelly–Mac Lane graph as the identity deduction on A, the two deductions cannot be equated. The identity deduction has characteristic 0, while this deduction has characteristic 1.

There are a number of basic equations an ADS must satisfy. They fall into five classes:

- Rules for making the connectives functorial.
- Rules for making the ADS a multi\polycategory, see [36, 46].
- Rules specified by the theory of closed categories, see [34, 2].
- Any added axioms must be natural in all variables.

This covers all the basic equations the ADS must satisfy. As already remarked, one may add additional equations subject to the above restriction. All of the basic equations do satisfy this property.

There is one further point to be made regarding equations. One of the advantages of classical linear logic is that the strong negation makes it possible to write one-sided sequents. However, we have been writing two-sided sequents in keeping with the category-theoretic tradition. A sequent of the form $\Gamma \vdash \Delta$ should be viewed as the corresponding one-sided sequent, $\vdash \Gamma^{\perp}, \Delta$. This can be viewed either as a consequence of an involutive negation in linear logic, or as a consequence of the adjoint properties of negation in a $*$-autonomous category.

On the other hand, all of the work in this paper can be reformulated in terms of one-sided sequents, in an evident way.

Note that the theories we are specifying do not include units. As pointed out previously, proof nets have not been extended to include the multiplicative units.

8. Addition of the MIX rule

There is also a limited amount of flexibility in adding inference rules. The only inference rule that can be added is the MIX rule. It can be added according to the following guidelines.

MIX can only be added to a classical ADS. (Note that MIX cannot be added in intuitionistic linear logic at all, since the intuitionistic sequents allow only one formula to the right of the turnstile.) The rule for adding MIX is as follows. If the ADS has no additional axioms or if each additional axiom is derivable in the theory mLL + (MIX), then MIX need not be added as an inference rule, but one is free to do so. If an axiom is added which is not derivable for the theory mLL + (MIX), then the MIX rule must be added as an inference rule. The reason for this will become clear in the proof of the compatibility theorem, Theorem 9.3. The MIX rule is necessary to construct the incompatible sequent.

Since this is a general system, it is important to know which monoidal closed theories can be specified using ADS. In fact, infinitely many theories can be specified in this way, including the following.

Examples. (1) In the intuitionistic case, with no additional axioms or equations, the result is the theory of autonomous categories; see [34].

(2) In the classical case, with no additional axioms or equations, the result is the theory of $*$-autonomous categories, due to Barr; see [2, 44].

(3) In the classical case, add in two additional axioms, each with the evident graph:

$$\boldsymbol{A} \wp \boldsymbol{B} \vdash \boldsymbol{A} \otimes \boldsymbol{B},$$

$$\boldsymbol{A} \otimes \boldsymbol{B} \vdash \boldsymbol{A} \wp \boldsymbol{B}.$$

Also, add the equations that make the two inverse to each other. Then the result is the theory of compact closed categories, due to Kelly and La Plaza [32, 33].

(4) Only adding the second of the two axioms can be thought of as adding a limited weakening to the theory. Girard's category of coherence spaces (see [17, 26]) provides a model. It is a consequence of the MIX rule mentioned in the introduction. So, this is an example of a sequent which is valid in mLL + (MIX), but not in mLL. Following a suggestion of Seely [44], we call these MIX-autonomous categories.

(5) Alternatively, we might only add the axiom

$$\boldsymbol{A} \wp \boldsymbol{B} \vdash \boldsymbol{A} \otimes \boldsymbol{B}$$

again with the evident graph. The recent work [9, 3] has given examples of this structure. These are the shift groups. They are defined as follows.

Given an ordered abelian group, such as $(\mathbf{Z}, +)$, define a $*$-autonomous structure by first choosing a fixed element, a, then defining:

$$X \wp Y = X + Y,$$

$$X \otimes Y = X + Y - a,$$

$$X^{\perp} = a - X.$$

It is straightforward to see that this is a $*$-autonomous poset. In the case of $(\mathbf{Z}, +)$, depending upon what value we choose for a, then the poset will have additional structure, as well. If a is chosen to be positive, then we have the structure of a MIX-autonomous category. If a is chosen to be 0, then it is compact-closed, since this amounts to equating $\otimes$ and $\wp$. If a is chosen negative, then we have the structure of Example 5. We call such categories COMIX autonomous.

We now define what it means for an ADS to satisfy composability and compatibility. Compatibility of graphs is defined earlier in the paper, and [34].

9. Compatibility for an ADS

Definition 9.1. Two sequents, $\Gamma \vdash \Delta$ and $\Gamma' \vdash \Delta'$ are said to be *composable* if the following holds. If we are in an ADS without the mix rule and $\Gamma', \Delta \neq \emptyset$ (if either is empty, they are not composable), then the sequents are composable if $\otimes\, \Gamma' = \wp\, \Delta$. Of course, for these two formulas to be equal, either Γ or Δ must have only one formula. If the ADS does have the mix rule, then the sequents are composable if either the above holds or both Γ', Δ are empty.

The intention of this definition is that when viewed categorically the morphisms interpreting the deduction should be composable. The fact that we have chosen to write the MIX rule as

$$\frac{\Gamma \vdash \qquad \vdash \Delta}{\Gamma \vdash \Delta}$$

necessitates the second part of the definition.

Definition 9.2. An ADS satisfies compatibility, or is *compatible*, if any two composable, derivable sequents have underlying graphs which are compatible.

Note. *The following theorems are all for classical ADS only. They can be modified to include the intuitionistic case. We will mention this modification later in the paper.*

Also note that the word *graph* is being used in two different ways; the Kelly–Mac Lane notion of graph, which corresponds to the axiom links, and the Danos–Regnier notion which corresponds to the entire proof structure. There should be no confusion, but we will refer to Kelly and Mac Lane's notion as simply "graphs", and the Danos–Regnier notion as "proof structures".

Theorem 9.3. *An ADS satisfies compatibility iff each axiom has a proof structure which is a generalized proof net. This means that it is a valid sequent in mLL + (MIX) and that its graph corresponds to the axiom links of a generalized proof net.*

The idea behind the proof is that any loop formed when attempting to compose the graphs will be a short trip, and vice versa.

Proof of Theorem 9.3. $\Leftarrow$: The proof is by contradiction. Suppose we have two sequents, $\Gamma \vdash \Delta$ and $\Gamma' \vdash \Delta'$ which are valid in mLL + (MIX), and which have graphs corresponding to correct deductions. Suppose also that they are composable and incompatible. If they are incompatible it follows that both Δ and Γ' are nonempty. So, replace Δ with $\wp\Delta$ and Γ' with $\otimes(\Gamma')$. Since the two sequents are composable, these two formulas must be equal. Call this formula $\boldsymbol{S}$. We, thus, have two sequents, $\Gamma \vdash \boldsymbol{S}$ and $\boldsymbol{S} \vdash \Delta'$, which are composable and incompatible. We will show that the following deduction is incorrect in mLL + (MIX)

$$\frac{\Gamma \vdash \boldsymbol{S} \qquad \boldsymbol{S} \vdash \Delta'}{\Gamma \vdash \Delta'}.$$

We will do so by showing that there is a cycle in the corresponding proof structure. Since all of the inference rules of the theory mLL + (MIX) preserve the correctness criterion of acyclicity, it will follow that the ADS must contain an axiom incorrect for that theory.

One way of proving the existence of a cycle in the proof structure of this deduction is via cut elimination. Since the cut formula $\boldsymbol{S}$ is a compound formula, then cut elimination proceeds by applying the first of the cut elimination procedures described in the section on proof nets. Continue this process until you arrive at the case where all the remaining cuts are atomic. Then the proof structure will have a substructure of the form shown in Fig. 9.

Since this proof structure clearly contains a cycle, the original structure must have, as well. For, the cut elimination procedure preserves correctness.

Thus, the ADS must contain a sequent incorrect in mLL + (MIX).

That concludes the proof of this direction of the theorem.

$\Rightarrow$: For this direction, we prove the contrapositive. Suppose that there exists a sequent $\Gamma \vdash \Delta$ which is not correct in mLL + (MIX). So, in particular, its cut-free proof structure has a cycle for some setting of the switches. We will construct a pair of composable deductions with incompatible graphs.

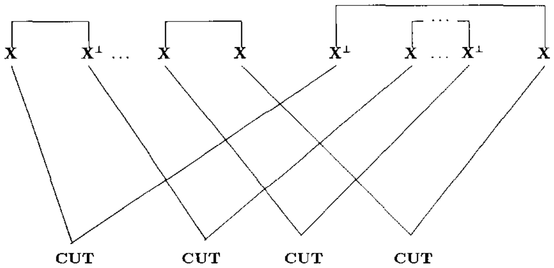

Fig. 9.

Choose one of the cycles appearing in the proof structure, and list all the vertices, i.e. the formulas, which appear in this cycle. Some of these formulas may arise in the sequent as subformulas of formulas in Γ. Bring any such formula in Γ to the other side of the sequent by using linear negation. This yields a new sequent $\Gamma' \vdash \tilde{\Delta}$. Clearly, $\tilde{\Delta}$ is nonempty and, so, may be replaced by $\boldsymbol{S} = \wp(\tilde{\Delta})$ by using the right $\wp$ rule. Thus, we have a new sequent $\Gamma' \vdash \boldsymbol{S}$. This will be the first of our two sequents. We construct a second sequent $\boldsymbol{S} \vdash \Delta'$ which is incompatible with $\Gamma' \vdash \boldsymbol{S}$.

We give an inductive procedure for constructing the sequent $\boldsymbol{S} \vdash \Delta'$. Begin by listing all the formulas appearing in the chosen cycle. The list should be written in the following order. Begin at one of the propositional atoms. Then proceed by following the path of the cycle. The second formula in the list should be the propositional atom to which the first formula is paired by an axiom link. This determines the direction to proceed around the cycle. Now continue in this direction until all the vertices have been listed. So, we obtain a sequence of formulas:

$$\boldsymbol{T}_1, \boldsymbol{T}_2, \ldots, \boldsymbol{T}_n.$$

Now, consider the subsequence Σ of just the propositional atoms:

$$\Sigma = \boldsymbol{A}_1, \boldsymbol{A}_1^{\perp}, \ldots, \boldsymbol{A}_m, \boldsymbol{A}_m^{\perp}$$

Because of the order in which the $\boldsymbol{T}_i$'s were listed. Σ must be of this form. We now make the following substitution in $\boldsymbol{S}$, where $\boldsymbol{A}$ is a new propositional atom: Let

$$\boldsymbol{A} = \boldsymbol{A}_1 = \boldsymbol{A}_2 = \cdots = \boldsymbol{A}_m.$$

Also, this forces the following equalities:

$$\boldsymbol{A}^{\perp} = \boldsymbol{A}_1^{\perp} = \cdots = \boldsymbol{A}_m^{\perp}$$

So, actually, we have modified $\boldsymbol{S}$ slightly and, thus, the sequent $\Gamma' \vdash \boldsymbol{S}$. But, this goes along with the intuition that is being developed that the axiom links are the real variables.

We first introduce a pairing of the variables in Σ which will correspond to axiom links for the sequent $\boldsymbol{S} \vdash \Delta'$.

The pairings are given by:

A_1 is paired to $A_m^\perp$,

A_2 is paired to $A_1^\perp$,

A_3 is paired to $A_2^\perp$,

$\vdots$

A_m is paired to $A_{m-1}^\perp$.

This motivates the above substitution in S.

The paired atoms will be introduced into the deduction of $S \vdash \Delta'$ via the identity rule, which for this deduction, and for these atoms, will be written as

$$A, A^\perp \vdash.$$

Note that we can do this because each of the A_i's has been equated to the new variable A.

The formula S may also contain subformulas which are not part of the loop. More precisely, S may have a subformula B which contains none of the atoms in the list Σ. If B is such a formula, it may be introduced into the deduction of $S \vdash \Delta'$ by the identity rule. For formulas such as this, the identity axiom is written as

$$B \vdash B.$$

The formula B should always be chosen to be the maximal such subformula.

This determines all of the instances of the identity rule which we will need for the deduction of $S \vdash \Delta'$. In other words, this determines all of the axiom links of the corresponding proof structure. All that remains is to establish how the connectives are introduced.

Since the sequent $S \vdash \Delta'$ and its axiom links are now determined, the cut-free proof structure is uniquely determined. It remains to verify that the corresponding deduction is correct, so that we may actually construct this sequent in the ADS.

The deduction is built inductively, based on the complexity of the proof structure. At each stage, it will be necessary to introduce one of the two connectives $\otimes$ or $\wp$.

If the connective to be introduced is a $\otimes$, then use the following deduction:

$$\frac{\dfrac{T_1 \vdash \Gamma_1 \qquad T_2 \vdash \Gamma_2}{T_1, T_2 \vdash \Gamma_1, \Gamma_2}}{T_1 \otimes T_2 \vdash \Gamma_1, \Gamma_2}.$$

This deduction uses the MIX rule followed by the left $\otimes$ rule.

If it is necessary to put a $\otimes$ link between two terms T_1, T_2 which are already in the same sequent, then just use the deduction

$$\frac{T_1, T_2 \vdash \Gamma}{T_1 \otimes T_2 \vdash \Gamma}.$$

If the connective to be introduced is a $\wp$, then the deduction proceeds as follows:

$$\frac{T_1 \vdash \Gamma_1 \qquad T_2 \vdash \Gamma_2}{T_1 \wp T_2 \vdash \Gamma_1, \Gamma_2}.$$

There is one more construction which may a priori be required. It is conceivable that, in the course of constructing S, one may need to put a $\wp$ between two terms already in the same sequent. This is incorrect in both mLL and mLL+(MIX).

But it is claimed that it is never necessary to do this. Since a $\wp$ link is switchable, it cannot be the case that, for a given $\wp$ link, both premises and the conclusion appear as part of the loop in the sequent $\Gamma' \vdash S$. So, either only one of the premises, or both the premises but not the conclusion, can be part of the loop. Since those formulas B which are not part of the loop are chosen to be maximal when they are introduced into the deduction, the possibility that neither of the premises is part of the loop is excluded.

In the first case, the construction required is to use the left $\wp$ rule to join a sequent of the form $B \vdash B$, where B is a formula not contained in the loop, to a sequent of the form $T \vdash \Gamma$, a sequent containing one of the A_i's.

In the second case, the two formulas T_1 and T_2 will still be in separate sequents when it is required to connect them with the $\wp$.

In either situation, the problem is avoided.

The construction now proceeds by induction on the complexity of S.

The induction process is well-defined. The structure of S uniquely determines which connective to be introduced and the two sequents to be joined by this connective.

We, thus, obtain a deduction of the sequent $S \vdash \Delta'$. Δ' will consist of the disjoint union of all those maximal formulas **B** which appear in S but do not contain any of the atoms in Σ. It is also clear by the construction what the Kelly–Mac Lane graph of this deduction is.

We have now constructed two composable sequents, $\Gamma' \vdash S$ and $S \vdash \Delta'$. It remains to show that they are incompatible.

To prove incompatibility, one must give a sequence of propositional atoms in S, and check that the Kelly–Mac Lane graph creates a loop through these atoms. That sequence will just be Σ. That a loop is formed follows from the fact that the original proof structure contained a cycle and that the sequent $S \vdash \Delta'$ was constructed based on this cycle. The path of the cycle determined how the axiom links were introduced. □

Intuitively, what has been done is to construct a generalized proof net from the original structure by the following transformation.

The cycle in the original proof structure must be of the basic shape (see Fig. 10a). This cycle is transformed into a structure of the form shown in Fig. 10b.

The $\wp$'s in this structure become tensors when brought to the other side of the sequent.

Notice the disconnected nature of this structure. That is why we had to work with generalized proof nets, and the MIX rule.

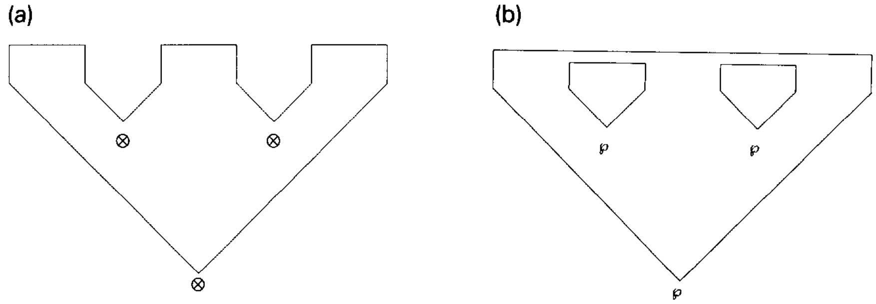

Fig. 10.

Those sequents $\Gamma' \vdash S$ for which the construction does not require the MIX rule will be of interest in the intuitionistic case; see Theorem 10.6.

We illustrate the process of constructing incompatible sequents with an example. Suppose that we have an ADS with the following additional axiom, with the evident graph:

$$B \vdash A \otimes (A^{\perp} \wp B)$$

The deduction of the incompatible sequent is as follows:

$$\cfrac{\cfrac{A, A^{\perp} \vdash \qquad B \vdash B}{A, A^{\perp} \wp B \vdash B}}{A \otimes (A^{\perp} \wp B) \vdash B}.$$

Essentially, the proof provides two inverse constructions. The first constructs a cycle from an instance of incompatibility, while the second constructs an instance of incompatibility from a cycle. This shows that the Kelly–Mac Lane concept and the Danos–Regnier concept are equivalent in a strong sense.

10. Coherence for an ADS

Definition 10.1. An ADS satisfies *coherence* if any two deductions of $\Gamma \vdash \Delta$ with the same graph are equal.

Theorem 10.2. *An ADS satisfies coherence iff each axiom corresponds to a generalized proof net.*

In other words, coherence = compatibility.

The main tool in proving this theorem is the confluent nature of the cut elimination process for proof nets. The unique normal form of the net obtained by this process provides a faithful representation of each morphism in the theory.

Proof of Theorem 10.2. $\Leftarrow$: Suppose that we have an ADS in which each axiom corresponds to a generalized proof net.

Given a set of deductions, each of the same sequent and with the same graph, we define a reduction system on the deductions which is confluent and strongly normalizing.

The reduction system is simply cut elimination for Danos–Regnier proof nets, as described previously. One must check that the reduction system preserves the equivalence relation on deductions. In other words, after the application of a reduction step, one still has the same deduction, up to the equivalence specified by the theory. This amounts to a straightforward case analysis.

As already remarked, this system is confluent and strongly normalizing; see [18].

It is straightforward to check that, given a valid sequent in mLL or mLL+(MIX) together with its axiom links, then the cut-free proof net is uniquely determined. The cut-free proof structure is given by what is essentially the subformula tree at each formula of the sequent together with its axiom links, i.e. its Kelly–Mac Lane graph.

Finally, one can show that the cut-free net, in addition to being the unique cut-free net with the given graph, represents a unique deduction, up to equivalence. In other words, it represents a unique morphism in the corresponding free category. This, combined with the confluence and strong normalization of the above reduction system, implies that every deduction of this sequent converges to the unique cut-free proof net. This direction of the theorem now follows. Since there is a unique normal form which every deduction of the given sequent converges to, they must all be equal.

$\Rightarrow$: We prove this direction by contradiction. By Theorem 9.3, it is sufficient to show that if an ADS has an instance of incompatibility, then it is not coherent. We first need the following lemma.

Lemma 10.3. *Suppose we have two sequents, $\Gamma \vdash \Delta$ and $\Gamma' \vdash \Delta'$ which are composable and have incompatible graphs, and let A be a variable which appears in one of the loops. Then the above two sequents can be replaced by two new sequents which are still composable and incompatible and in which the variable A appears only in the instances which are part of the loop.*

Proof of Lemma 10.3. Let A be a variable appearing in a loop. Suppose also that A appears elsewhere in the deduction other than in the loop. It must be the case that this occurrence of the variable is not paired by the Kelly–Mac Lane graph with the occurrence which is part of the loop. For, if a variable is part of a loop, so is its mate. Thus, this deduction is derived by substitution from a more general deduction for which these two occurrences are replaced by distinct variables. This follows from the convention on additional axioms. The lemma now follows.

Proof of Theorem 10.2 (*continued*). The proof of the theorem now proceeds as follows. We must construct two nonequivalent deductions of the same sequent, with the same Kelly–Mac Lane graph. By assumption, there are two sequents $T \vdash S$ and $S \vdash R$ which are composable and incompatible. Without loss of generality, we have replaced the sequents with their categorical counterparts. So, T and R are either single formulas or empty. S is a single formula. Now, let A be one of the variables which appears in the loop. Use the above lemma to replace any other occurrences of A which do not appear in the loop by a different, new variable. So, we have two new sequents $T' \vdash S'$ and $S' \vdash R'$ which are still composable and incompatible. Now, choose two new variables, say C and D. Substitute $C \otimes D$ for A. This yields two new sequents $T'' \vdash S''$ and $S'' \vdash R''$. However, it is the case that T'' is equal to T', and that R'' is equal to R'. This follows from the fact that we have replaced any occurrences of A not appearing in the loop by a fresh variable. So, in particular, it no longer occurs in either T' or R'. Thus, we have two deductions of the following form:

DEDUCTION $\mathscr{D}_1$

$$\frac{T' \vdash S' \qquad S' \vdash R'}{T' \vdash R'}.$$

and

DEDUCTION $\mathscr{D}_2$

$$\frac{T' \vdash S'' \qquad S'' \vdash R'}{T' \vdash R'}.$$

So, we have two deductions of the sequent $T' \vdash R'$. First, it is claimed that these two deductions assign the same graph to the sequent. This is because the only difference between the two deductions occurs in one of the loops formed by graph composition. But, as already remarked, graph composition ignores any loops which are formed. It remains to show that these two deductions are not equivalent. This will follow by calculating the characteristics of the two deductions. It is the case that

$$\chi(\mathscr{D}_2) = \chi(\mathscr{D}_1) + 1.$$

This is because the introduction of the new term $C \otimes D$ serves to introduce one additional loop being formed when you compose the two graphs of deduction $\mathscr{D}_2$. Since one of the conditions for when two deductions can be equated is that they have equal characteristic, then $\mathscr{D}_1$ is not equivalent to $\mathscr{D}_2$, and we are done. $\square$

Intuitionistic case:

For the intuitionistic case, the theorems need to be modified slightly. Since sequents are of the form $\Gamma \vdash A$, where A is a single formula, the MIX rule cannot even be stated in intuitionistic mLL (ImLL). Also, since the construction in the proof of the compatibility theorem explicitly uses the MIX rule, it is invalid for ImLL.

Definition 10.4. An ImLL sequent is *strongly incompatible* if the following holds. When the sequent is viewed as a classical sequent (so change $A \multimap B$ to $A^{\perp} \wp B$), it is incorrect in the theory mLL + (MIX) and the construction of incompatible sequents in the proof of the compatibility theorem does not require the MIX rule.

In [5], there is an intrinsic characterization of strongly incompatible sequents. It is not known what happens if there is an incompatible sequent which is not strongly incompatible. However, in the intuitionistic case, there seem to be no examples which come up in practice of intuitionistic ADS, except for the one with no additional axioms. Theorems 9.3 and 10.2 above become four theorems for intuitionistic ADS, as follows.

Theorem 10.5. *For an intuitionistic ADS, if all of the axioms, when viewed as classical sequents are valid in mLL + (MIX), then the ADS satisfies compatibility.*

Theorem 10.6. *If the ADS contains a strongly incompatible sequent, then the ADS is incompatible.*

Theorem 10.7. *If all of the axioms of the ADS are valid in ImLL, then the ADS is coherent.*

Theorem 10.8. *If the ADS is coherent, it is compatible.*

Proofs. Same as in the classical case. □

Corollary 10.9. Table 1 presents a corollary of the above theorems.

The proof is by examination. In the compact-closed case, the first of the two axioms fails in mLL + (MIX). This is easiest to see using proof nets. In the case of MIX-autonomous categories, it can be verified that this axiom is a consequence of the mix rule. So, the mix rule implies that $\otimes$ implies $\wp$, which can be thought of as a form of weakening. The deduction is as follows:

$$\dfrac{\dfrac{\dfrac{\boldsymbol{A} \vdash \boldsymbol{A} \qquad \boldsymbol{B} \vdash \boldsymbol{B}}{\boldsymbol{A}, \boldsymbol{B} \vdash \boldsymbol{A}, \boldsymbol{B}}}{\boldsymbol{A}, \boldsymbol{B} \vdash \boldsymbol{A} \wp \boldsymbol{B}}}{\boldsymbol{A} \otimes \boldsymbol{B} \vdash \boldsymbol{A} \wp \boldsymbol{B}}.$$

The first of the above five cases is analogous to the Kelly–Mac Lane result, but not quite as strong. What the Kelly–Mac Lane result says is that coherence holds in the theory of autonomous categories for all allowable morphisms which do not contain subshapes of the form $\boldsymbol{R} \multimap \boldsymbol{I}$; see [34]. This corollary says coherence holds in the theory of autonomous categories without units. However, this approach has the advantage that it allows one to study coherence for autonomous categories with additional structure, i.e. additional morphisms or equations.

Table 1

Theory	Compatibility	Coherence
Autonomous	Yes	Yes
$*$-Autonomous	Yes	Yes
Compact-closed	No	No
MIX-autonomous	Yes	Yes
COMIX-autonomous	No	No

Using ADS, it is possible to give a straightforward criterion for determining whether there is a morphism of a given graph. Simply write down the corresponding cut-free proof structure, and determine if it is a net. Kelly and Mac Lane also have such a criterion, which can be derived from their cut elimination theorem, the "allowable = constructible" lemma. However, this criterion requires a certain amount of case analysis which proof nets avoid. The "allowable = constructible" lemma that every deduction reduces to onc of five normal forms. Using proof nets, we are able to obtain a unique normal form.

11. Models of ADS

All the work done up to this point has been entirely at the level of the theories given by ADS. But the dinaturality properties which we will now study can be viewed at the level of models as well. So, we now discuss the model theory of an ADS.

Autonomous deductive systems were defined with the idea that they would be logical systems which would have as models the particular closed categories being specified, i.e. the compact closed categories, etc. We now outline the extent to which this holds. For more details, see [5].

Since we are modeling sequents, the most obvious notion of the model of an ADS is via multicategories or polycategories; see [36, 46]. These are a generalization of category to allow finite lists of objects to appear in either the domain or codomain. The notion of $*$-autonomous multicategory or $*$-autonomous polycategory can be defined in the evident way. This gives the simplest way of discussing models. However, to work in ordinary categories, we proceed as follows.

There is a slight distinction between the presentation of mLL and that of $*$-autonomous categories. In the former, the primitives are taken to be $\otimes$ and $\wp$, and negation is built into the syntax. In the latter, the connectives are $\otimes$ and negation. $\wp$ is a defined operation. This is not a problem if we assume double negation to be strict, in other words, that

$$(-)^{\perp\perp} = identity.$$

However, this restriction will be unnecessary. It turns out that this is not a significant problem, since our sequents contain only formulas of the form A and $A^{\perp}$, and since there is a *coherent* isomorphism between A and $A^{\perp\perp}$ in an arbitrary $*$-autonomous category, we do not need to make this restriction.

Even though we do not have units in our theory, they will still be necessary in the models in that, for both the classical and intuitionistic cases, it is possible to have one of the two sides of the turnstile empty. We need units to model this categorically. For example, the intuitionistic sequent

$$\vdash A$$

is interpreted categorically as

$$I \to A.$$

So, we will need units to a certain extent. So, we introduce as part of the definition of model that there are units I and $I^{\perp}$. But this is with the understanding that our coherence applies only to morphisms where the unit appears only as the sole formula of the domain or codomain.

So, the interpretation of classical sequents proceeds as follows.
The sequent, $\vdash \Delta$ is interpreted as

$$I \to \wp(\Delta).$$

The sequent $\Gamma \vdash$ is interpreted as

$$\otimes(\Gamma) \to I^{\perp}.$$

Sequents of the form $\Gamma \vdash \Delta$, where Γ and Δ are nonempty are modelled by morphisms of the form

$$\otimes\, \Gamma \to \wp(\Delta).$$

To model the MIX rule, we add a canonical morphism

$$I^{\perp} \to I.$$

The MIX rule is then modelled by the composite

$$\otimes\, \Gamma \to I^{\perp} \to I \to \wp\Delta.$$

This is the reason for writing the MIX rule in the way mentioned earlier.

12. Dinaturality

Like the subject of coherence theorems, dinatural transformations arose in the area of algebraic topology. Recently, they have been of great use in modelling polymorphism; see [1, 27]. In the remainder of the paper, we explore the connections between coherence theorems and the composition problem for dinatural transformations. It will be the case that any ADS naturally gives rise to a class of dinatural transformations and that, as a consequence of coherence, these dinaturals have several important properties.

Polymorphic types are a generalization of simple types to include type variables. This allows one to introduce a certain amount of uniformity into functions, such as the identity function, which are defined at every type. For a more complete discussion, see [1].

To model polymorphic types, a first attempt would be to have types represented by functors. However, this approach cannot be implemented in any evident way. Some of the expressions which most frequently arise in polymorphism cannot be interpreted directly as functors.

For example, the type expressions,

$$X \otimes X^{\perp} \quad \text{and} \quad X \multimap X$$

cannot be viewed as functors, since the variable X appears in both a covariant and a contravariant position. There are several possible solutions as outlined in [1].

The solution we examine here is to introduce a special class of multivariant functors to represent the types. So, if σ is a type, its interpretation, denoted as $|\sigma|$ will be a functor

$$|\sigma| : (C^{\mathrm{op}})^n \times C^n \to C,$$

where C is a model of an ADS. The interpretation is built inductively, as follows.

Let σ be a proposition, its type variables contained in the list $X_1, \ldots, X_n$. Its interpretation $|\sigma|$ is defined as follows:

If $\sigma = X_i$, $|\sigma|$ is the ith covariant projection.

If $\sigma = X_i^{\perp}$, then $|\sigma|$ is the ith contravariant projection.

If $\sigma = \tau_1 \otimes \tau_2$, $|\sigma| = |\tau_1| \otimes |\tau_2|$.

If $\sigma = \tau_1 \wp \tau_2$, then $|\sigma| = |\tau_1| \wp |\tau_2|$.

As a consequence of this, if $\sigma = \tau^{\perp}$, $|\sigma|(\bar{A}, \bar{B}) = (|\tau|(\bar{B}, \bar{A}))^{\perp}$.

Since $(\ )^{\perp}$ is contravariant, the variance of each variable switches, just as it does in the "twisted" exponential in [1, 27].

Linear logic seems particularly well suited to this approach because the atoms are introduced in pairs as A and $A^{\perp}$. This naturally leads to this sort of multivariant notion of semantics.

In this approach, it is clear that ordinary natural transformations are not adequate to model typing judgements, in other words, the terms. We use instead dinatural transformations.

Definition 12.1. Given two functors $F, G : (C^{\mathrm{op}})^n \times C^n \to C$, a *dinatural transformation* is a family of morphisms

$$\{\Theta = \Theta_A : FAA \to GAA \mid A \text{ in } C^n\}$$

such that, for all $f : A \to B$ in C^n, the diagram in Fig. 11 commutes.

$$\begin{tikzcd}
& FAA \arrow[r, "\theta_A"] & GAA \arrow[dr, "GAf"] & \\
FBA \arrow[ur, "FfA"] \arrow[dr, "FBf"'] & & & GAB \\
& FBB \arrow[r, "\theta_B"] & GBB \arrow[ur, "GfB"'] &
\end{tikzcd}$$

Fig. 11.

It will be the case that certain deductions in certain ADS can naturally be interpreted as dinatural transformations, as we will show below.

We will assume we are in a model of the ADS with no axioms or equations, for the moment. So, we are in a $*$-autonomous category. Previously, deductions were viewed as yielding single morphisms. Here we show how deductions actually yield a family of morphisms, indexed by the objects of $\boldsymbol{C}^n$. We will then explore to what extent this family will be dinatural.

More specifically, given a cut free deduction of a sequent

$$\Gamma \vdash \Delta \quad (\Gamma, \Delta \textit{ nonempty})$$

which has its propositional atoms among the list

$$\{A_1, \ldots, A_n\},$$

we will replace these atoms by type variables

$$\{\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n\}.$$

Then we will apply our type interpretation to each of the propositions in Γ or Δ. Then there will be a family of morphisms which will be dinatural between the following two multivariant functors:

$$\bigotimes |\Gamma| \to \wp |\Delta|.$$

If Γ is empty, the above yields a dinatural family

$$\boldsymbol{K}_I \to \wp |\Delta|,$$

where $\boldsymbol{K}_A$ is the constant functor valued at A. If Δ is empty, we get a family

$$\bigotimes |\Gamma| \to \boldsymbol{K}_{I^\perp}.$$

The construction is built inductively on the length of the deduction.

We are assuming for the moment that there are no added axioms in the ADS. If the deduction is the identity

$$\boldsymbol{A} \vdash \boldsymbol{A}$$

then we pick as the family, the family of identity morphisms, indexed by the objects of $\boldsymbol{C}$.

If the following two deductions

$$\Gamma \vdash \Delta, \boldsymbol{A} \qquad \Gamma' \vdash \Delta', \boldsymbol{B}$$

yield families, then we get a family of morphisms corresponding to the deduction

$$\frac{\Gamma \vdash \Delta, \boldsymbol{A} \qquad \Gamma' \vdash \Delta', \boldsymbol{B}}{\Gamma, \Gamma' \vdash \Delta, \Delta', \boldsymbol{A} \otimes \boldsymbol{B}}.$$

This is obtained by using the fact that our category is a model and, so, has the appropriate operations to construct such a family.

We, thus, get a family of morphisms of the form

$$\otimes(\Gamma, \Gamma') \rightarrow \wp(\Delta, \Delta', \boldsymbol{A} \otimes \boldsymbol{B}).$$

The rules $\otimes \boldsymbol{L}$, $\wp \boldsymbol{R}$, $\wp \boldsymbol{L}$ and negation are treated analogously.

Of course, we must assume in the above construction that the list of variables $\{\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n\}$ contains all the variables in both sequents.

Theorem 12.2. *Each of the families obtained by the above construction is dinatural.*

Proof. Each of the above constructions involves only the following categorical operations:
- application of functors,
- composition with ordinary natural transformations, for example, the symmetry, or associativity isomorphisms,
- application of the transpose operation (evaluation or coevaluation).

It is a standard result that all the above procedures preserve dinaturality. For example, the second preserves dinaturality by the diagram shown in Fig. 12. If ζ is an ordinary natural transformation from E to F, then this diagram clearly commutes. For a reference for these results, see [45, 27].

$$\begin{tikzcd}
 & EAA \arrow[r, "\zeta_{AA}"] & FAA \arrow[r, "\theta_A"] & GAA \arrow[dr, "GAf"] & \\
EBA \arrow[ur, "EfA"] \arrow[r, "\zeta_{BA}"] \arrow[dr, "EBf"'] & FBA \arrow[ur, "FfA"] \arrow[dr, "FBf"] & & & GAB \\
 & EBB \arrow[r, "\zeta_{BB}"] & FBB \arrow[r, "\theta_B"] & GBB \arrow[ur, "GfB"] &
\end{tikzcd}$$

Fig. 12.

The problem with this approach is in modelling the CUT rule. The CUT rule should be modelled by composing the dinaturals. However, it is well known that dinatural transformations need not compose. To compose them, you would hope that the diagram in Fig. 13 commutes, where ι is a dinatural transformation from G to H.

However, even if the two inner hexagons compose, the outer hexagon may not. An example will be given, in a moment.

Thus, to model polymorphic types effectively, we need to do more than give a class of dinaturals. This class should be a class which composes.

As the following example shows, the issue of composability is closely related to compatibility and, hence, to coherence. The relation to compatibility was first noted in [14].

Example. This example can be found in [1]. We give two dinatural transformations in the category of sets which do not compose. The three functors involved are

$$\boldsymbol{K}_1, (\ )^{(\ )}, \boldsymbol{K}_{BOOL} : \boldsymbol{C}^{\mathrm{op}} \times \boldsymbol{C} \to \boldsymbol{C}.$$

The dinaturals

$$\theta : \boldsymbol{K}_1 \to (\ )^{(\ )}$$

and

$$\eta : (\ )^{(\ )} \to \boldsymbol{K}_{BOOL}.$$

are described as follows. For θ, θ_A is the transpose of the identity on A.

$$\theta_A = \ulcorner id \urcorner : 1 \to A^A$$

For $\eta, \eta_A : A^A \to BOOL$ is given by the following definition. Let f be an endomorphism on A.

$$\eta_A(f) = \begin{matrix} T & \text{if } \# \text{ of fixed points of } f \text{ is even or } \infty, \\ F & \text{otherwise.} \end{matrix}$$

Both of the above families are easily seen to be dinatural.

$$\begin{tikzcd}
 & FAA \arrow[r, "\theta_A"] & GAA \arrow[r, "\iota_A"] \arrow[dr, "GAf"] & HAA \arrow[dr, "HAf"] & \\
FBA \arrow[ur, "FfA"] \arrow[dr, "FBf"'] & GBA \arrow[ur, "GfA"] \arrow[dr, "GBf"'] & & GAB & HAB \\
 & FBB \arrow[r, "\theta_B"'] & GBB \arrow[r, "\iota_B"'] \arrow[ur, "GfB"'] & HBB \arrow[ur, "HfB"'] &
\end{tikzcd}$$

Fig. 13.

Their composition should be a dinatural family from K_1 to K_{BOOL}. It is easily seen that a dinatural family between constant functors should be a single morphism. However, we actually get two morphisms when we compose the above two families, the composition depends on A:

$$1 \to A^A \to BOOL.$$

In particular, if A is even, then 1 gets mapped to T, and if A is odd, 1 gets mapped to F. So, the composition of the two families is not dinatural.

Intuitively, this is caused by the presence of the variable A, which does not appear after the composition. If one was to assign graphs to each of the two dinaturals, they would be as follows. To see how the graph is obtained, consider the two placeholders, the (–)'s, as variables (see Fig. 14).

In short, they would be incompatible. Incompatibility of graphs allows "false dependencies", morphisms which depend on variables which do not appear after the composition. So, from this example and [14], there is good reason to believe that, for categories where the morphisms can be assigned graphs, compatibility may be the main issue. So, the dinatural families defined earlier can be studied from this viewpoint.

Of course, by previous theorems, compatibility is equivalent to coherence and, so, we hope to obtain a theorem relating dinaturality to coherence. This is the idea we now pursue.

We show that, for certain ADS, the dinaturals arising from the previous theorem actually compose and, thus, form a category. The key to providing this will be the previous coherence theorems.

Theorem 12.3. *Given the ADS with no additional axioms or equations, and without the MIX rule, and* **C**, *a model for this ADS, the construction of Theorem* 12.2 *yields syntactic families of morphisms which are dinatural and form a category. So, in particular, they compose. The category so constructed is also a model of the ADS.*

Note. The reason that these families should be viewed syntactically is that, in specific models, there may be certain identifications which allow for additional

$$\eta \circ \theta : 1 \longrightarrow (\ )^{(\ )} \longrightarrow BOOL$$

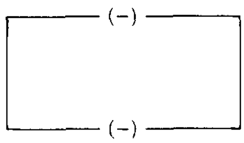

Fig. 14.

composition. But we are only interested in those pairs of families which are syntactically composable.

Proof. We will work in the classical case, the intuitionistic case is similar.

Suppose that we have two cut-free deductions which yield, by the previous theorem, dinatural transformations of the form

$$s: \boldsymbol{F} \to \boldsymbol{G},$$

$$t: \boldsymbol{G} \to \boldsymbol{H},$$

where $\boldsymbol{F}$, $\boldsymbol{G}$ and $\boldsymbol{H}$ are definable functors. Then, for each $\boldsymbol{A}$ in $\boldsymbol{C}^n$, there are morphisms of the form

$$s_A: \boldsymbol{FAA} \to \boldsymbol{GAA},$$

$$t_A: \boldsymbol{GAA} \to \boldsymbol{HAA}.$$

This yields, for each $\boldsymbol{A}$ in $\boldsymbol{C}^n$, a morphism $(\boldsymbol{t} \circ \boldsymbol{s})_A$. We must show this family to be dinatural.

Since s and t arise from deductions in mLL, there are cut-free deductions

$$\Gamma \vdash \Delta, \qquad \Gamma' \vdash \Delta'$$

such that these deductions yield s and t. Since these two deductions are composable, and we are in an ADS without the MIX rule, it must be the case that Δ and Γ' are both nonempty and that

$$\bigotimes \Gamma' = \wp \Delta.$$

We call this formula $\boldsymbol{G}$. So, now there are deductions

$$\Gamma \vdash \boldsymbol{G}, \qquad \boldsymbol{G} \vdash \Delta'.$$

By using the cut rule, there is a deduction of the form

$$\frac{\Gamma \vdash \boldsymbol{G} \qquad \boldsymbol{G} \vdash \Delta'}{\Gamma \vdash \Delta'}.$$

By cut elimination, there is a cut-free deduction of the sequent $\Gamma \vdash \Delta'$, with the same graph as the above deduction. Thus, by the previous theorem, there is a dinatural family, θ, corresponding to this deduction

$$\Theta: \bigotimes |\Gamma| \to \wp |\Delta'|.$$

It is also clear from the above that

$$\boldsymbol{F} = \bigotimes |\Gamma|, \qquad \boldsymbol{H} = \wp |\Delta'|.$$

If we can show that, for every $\boldsymbol{A}$,

$$\theta_A = t_A \circ s_A,$$

we will have shown that the family $\boldsymbol{t} \circ \boldsymbol{s}$ is, in fact, dinatural.

It is easily checked that, for every A, the graph of θ_A is the same as the graph of $t_A \circ s_A$. Since, by coherence, there is only one morphism of any given graph, it must be the case that

$$\theta_A = t_A \circ s_A$$

and we are done. $\square$

Note. The proof in the intuitionistic case is analogous.

A remark on the proof is in order. This proof works in an arbitrary model of this ADS. If we restrict to the free model, then we no longer require the intermediate step of cut elimination. The argument is as follows. Suppose we are in the free model, i.e. the free $*$-autonomous multicategory. It is the case that every morphism in this category has a Kelly–Mac Lane graph. So, consider the diagram in Fig. 15.

It will be the case that not only do the dinatural components have Kelly–Mac Lane graphs, but the morphisms FfA, GfA, GAf, etc., also have Kelly–Mac Lane graphs. This is not the case in an arbitrary model. It can then be verified that both of the morphisms from FBA to HAB in the diagram have the same Kelly–Mac Lane graph. It follows immediately from coherence that the diagram commutes. So, in this case, we get a simpler proof, not relying on cut elimination.

Next we address the problem of adding the MIX rule to the previous two theorems. Remember that, for our purposes, we are writing the MIX rule as

$$\frac{\Gamma \vdash \qquad \vdash \Delta}{\Gamma \vdash \Delta}.$$

In models of an ADS with the MIX rule, there is a canonical morphism:

$$I^{\perp} \to I,$$

$$\begin{tikzcd}
& FAA \arrow[r, "\theta_A"] & GAA \arrow[r, "\iota_A"] \arrow[dr, "GAf"] & HAA \arrow[dr, "HAf"] & \\
FBA \arrow[ur, "FfA"] \arrow[dr, "FBf"'] & GBA \arrow[ur, "GfA"] \arrow[dr, "GBf"'] & & GAB & HAB \\
& FBB \arrow[r, "\theta_B"] & GBB \arrow[r, "\iota_B"] \arrow[ur, "GfB"] & HBB \arrow[ur, "HfB"'] &
\end{tikzcd}$$

Fig. 15.

which we now view as a natural transformation

$$K_{I^\perp} \to K_I .$$

So, given deductions

$$\Gamma \vdash , \qquad \vdash \Delta ,$$

we extend the above dinatural interpretation by the following composition:

$$\bigotimes |\Gamma| \to K_{I^\perp} \to K_I \to \wp |\Delta| .$$

That this works is expressed by the following theorem.

Theorem 12.4. *Sequents of the form $\Gamma \vdash$ and $\vdash \Delta$ yield wedges and cowedges, respectively. The two previous theorems can be extended to the classical ADS with the MIX rule added by using the method just outlined.*

Note. The definition of wedge and cowedge can be found in [39].

Proof. That the MIX rule also yields dinaturals is seen from the diagram in Fig. 16. This diagram clearly commutes. The proof of the second theorem is still correct in the presence of the MIX rule, since the coherence theorem is still correct with the MIX rule added. □

The proof of these theorems is analogous to the proof in [27]. In that paper, the focus was on dinaturals arising from cartesian-closed categories. They set up a logical system, which is essentially intuitionistic logic, with $\wedge$ and $\Rightarrow$ as connectives, and with only exchange as a structural rule. This system corresponds syntactically to typed λ-calculus with product types. Cut elimination corresponds to β-reduction. It is well known that this reduction system is confluent and strongly normalizing; see [28, 38].

The general approach of using cut elimination, though, goes back to Lambek [36, 37]. These results suggest an axiomatic approach, based on reduction systems which we hope to explore at a later date.

$$\begin{tikzcd}
 & FAA \arrow[dr] & & & GAA \arrow[dr] & \\
FBA \arrow[ur] \arrow[dr] & & I^{\perp} \arrow[r] & I \arrow[ur] \arrow[dr] & & GAB \\
 & FBB \arrow[ur] & & & GBB \arrow[ur] &
\end{tikzcd}$$

Fig. 16.

The next issue is the extent to which other ADS, with other additional axioms or equations, satisfy dinaturality. In such an ADS, not all deductions will yield dinatural transformations. This may not even hold for cut-free deductions. It is too much to expect since there is a great deal of latitude in the axioms one is allowed to add.

As this is the case, the study of dinaturality in other ADS can best be accomplished by looking at certain specific models. We introduce a new category which will be a variation on the category of graphs. Remember that in the definition of graph composition, any loops formed when splicing together the links were ignored. Here, we choose to keep track of the loops. Define a category $\boldsymbol{G_L}$ in the following way. Its objects will be signed sets. Morphisms will be Kelly–Mac Lane graphs together with a disjoint union of loops. Composition will be graph composition, except that the loops associated with the composite will be the union of all the original loops together with any new loops formed.

Alternatively, a morphism in $\boldsymbol{G_L}$ could be defined to be a Kelly–Mac Lane graph together with a nonnegative integer which would correspond to its characteristic.

Theorem 12.5. *$\boldsymbol{G_L}$ is a model of every ADS. Given an ADS with arbitrary axioms and equations, cut free deductions yield dinatural transformations in $\boldsymbol{G_L}$.*

Proof. The proof of the first statement is straightforward. The equality rules and the conditions under which deductions may be equated is clearly satisfied by $\boldsymbol{G_L}$.

The second statement can be proved by induction on the complexity of the sequent.

Theorem 12.5 will hold only for cut free derivations. Once CUT is added, instances of incompatibility can occur, and the result no longer holds.

Theorem 12.6. *If the ADS has a nonlogical axiom which is not correct in $mLL+(MIX)$, then the dinaturals of the previous theorem do not compose.*

Proof. If the ADS has such an axiom, it follows from Theorem 9.3 that there is an instance of incompatibility. Thus, there are two deductions in the ADS

$$\boldsymbol{F} \vdash \boldsymbol{G} \quad \text{and} \quad \boldsymbol{G} \vdash \boldsymbol{H}$$

with underlying graphs which are incompatible. As in the proof of Theorem 9.3, $\boldsymbol{G}$ must be an actual formula and not the empty list. $\boldsymbol{F}$ and $\boldsymbol{H}$ may or may not be empty.

The method we use is analogous to that used in the proof of Theorem 10.2.

Let X be a variable whose vertex appears in the loop caused by incompatibility. Using the lemma from the proof of the coherence theorem, replace $\boldsymbol{F}, \boldsymbol{G}$ and $\boldsymbol{H}$ by $\boldsymbol{F}', \boldsymbol{G}', \boldsymbol{H}'$ to get two new deductions such that the variable X only appears as this instance of incompatibility.

This yields a morphism in $\boldsymbol{G}_L$ which will be of the form

$$\iota_A \circ \theta_A : FAA \to GAA \to HAA$$

Now choose a new variable, say V. In the above deduction, replace X by $X \otimes V \otimes V^{\perp}$. This yields a new morphism of the form

$$\iota_B \circ \theta_B : FBB \to GBB \to HBB$$

Now, there is an evident graph $f: A \to B$ in $\boldsymbol{G}_L$. It is obtained by connecting the occurrence of X in A with the corresponding occurrence in B. V is connected to the corresponding $V^{\perp}$. The graph is the identity on all other variables.

We will now show that the diagram in Fig. 17 does not commute. Since we have chosen deductions such that the variable X appears only in the shape $\boldsymbol{G}$, then X is a dummy variable in the shapes $\boldsymbol{F}$ and $\boldsymbol{H}$. Thus, the graphs FfA, HAf, FBf and HfB are all just identity graphs. By the same argument, it is clear that $FAA = FBB$ and that $HAA = HBB$. The morphisms

$$\iota_A \circ \theta_A : FAA \to GAA \to HAA,$$

$$\iota_B \circ \theta_B : FBB \to GBB \to HBB,$$

thus, have the same domain and codomain, but are not equal. The second morphism has two more loops than the first one. It, thus, follows that the diagram does not commute.

Thus, the composition is not dinatural. □

The reason why the proof works is that equality of morphisms in $\boldsymbol{G}_L$ represents the characteristic of the corresponding deduction, while equality in $\boldsymbol{G}$ does not.

What the two previous theorems show is that the previous results are best possible, that to extend an ADS beyond mLL+(MIX), one automatically loses dinaturality. A consequence of this is that if an ADS is dinatural, it must be contained in mLL+(MIX), and, so, be coherent.

$$\begin{tikzcd}
 & FAA \arrow[r, "\theta_A"] & GAA \arrow[r, "\iota_A"] \arrow[dr, "GAf"] & HAA \arrow[dr, "HAf"] & \\
FBA \arrow[ur, "FfA"] \arrow[dr, "FBf"'] & GBA \arrow[ur, "GfA"] \arrow[dr, "GBf"'] & & GAB & HAB \\
 & FBB \arrow[r, "\theta_B"] & GBB \arrow[r, "\iota_B"] \arrow[ur, "GfB"] & HBB \arrow[ur, "HfB"'] &
\end{tikzcd}$$

Fig. 17.

So, the two previous theorems say that dinaturality implies coherence.

Summing up the previous theorems, we claim the following theorem.

Theorem 12.7. *An ADS is dinatural, that is, the syntactic dinatural transformations compose in every model, iff it is coherent.*

We have, thus, established the equivalence of the three basic concepts of the paper, compatibility, coherence and dinaturality. The logical theory which corresponds to this equivalence is the theory mLL + (MIX). This suggests that this is the right level of generality to study monoidal closed categories.

13. Conclusion

There are several possibilities for further study, which we hope to explore. The definition of ADS begins with mLL as a base theory and then allows various extensions. But it would be desirable to study various weaker systems as well. An example would be the weakly distributive categories of [9]. Another benefit of studying weaker systems from this viewpoint is that it may be possible to give logical interpretations of such things as the Joyal–Street tensor calculus [31]. These possibilities are currently being explored in [7].

It is also possible to define a fibered version of ADS. Recently, Girard [21, 25], has extended proof nets to predicate mLL with quantifiers. The categorical counterpart would be a fibered category in which the pullback functors have assigned left and right adjoints. We can then obtain a fibered notion of coherence and a fibered analogue of dinatural transformation. These issues are explored in [6].

One of the interesting aspects of the dinatural interpretation is that, since the definition of dinatural transformation is equational in nature, this sort of model automatically comes equipped with a certain amount of parametricity. Thus, the interpretation will be "uniform" at all types, in the sense discussed in [42].

Along these lines, we can also use coherence to study the problem of determining the terms of a given type. The idea is to use the notion of nonstandard syntax discussed earlier to replace traditional terms ordinarily assigned to polymorphic types. So, now our terms are Kelly–Mac Lane graphs. The usual typing judgement system is replaced with proof nets. Then, to determine the number of terms of a given type, write down all the possible Kelly–Mac Lane graphs. Then determine which ones correspond to proof nets. By the coherence theorem, these are the only possible terms. So, for example, this method could be used to show that there are exactly two terms of type:

$$A \otimes A \multimap A \otimes A.$$

These, of course, correspond to the exchange map and the identity. All of the above seems to suggest that coherence may provide the correct categorical analogue of parametricity.

In another direction, theories over mLL have been of use recently in specifying concurrent systems, such as Petri nets; see [16, 40]. Since ADS are such theories, it would be of interest to study the systems which they specify. While they are less general than arbitrary mLL theories, the additional structure may have some computational meaning. For example, incompatibility of graphs should be a form of deadlock. In fact, if a pairing of a Kelly–Mac Lane graph is thought of as an interdependence relation, then the definition of incompatibility is precisely equivalent to Girard's definition of deadlock [22].

This is analogous to the results of Lafont [35] for interaction nets, where he shows that the presence of a cycle in an interaction net leads to deadlock in the corresponding program. This is not surprising since proof nets are the inspiration for interaction nets. The reduction rules of his nets behave in much the same way as the cut elimination procedure outlined above.

Finally, it is hoped that the structure presented can be thought of as a first version of "modular proof theory". Girard [19] points out the lack of modularity in traditional proof theory. He observes that slight alterations in the definition of the system necessitate that all the proof-theoretic results, such as cut elimination, be reproved. Given two systems which differ slightly in axioms or inference rules, there is generally no way to derive proof-theoretic results for one as a corollary of the other. What is desired is a general framework in which additional axioms and inference rules can be added, and any proof-theoretic results need be verified only on the additional structure. While ADS is clearly not as general as would be required by such a theory, its structure suggests several generalizations.

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